Developing Measures of Teachers' Mathematics Knowledge for Teaching

Heather Hill
Deborah Ball
Stephen Schilling
University of Michigan

Abstract

In this article, we discuss efforts to design and empirically test measures of teachers' content knowledge for teaching elementary mathematics. We begin by reviewing the literature on teacher knowledge, taking special note of how scholars have organized such knowledge. Next we describe survey items we wrote to represent knowledge for teaching mathematics and results from factor analysis and scaling work with these items. We found that teachers' knowledge for teaching elementary mathematics is multidimensional, and includes knowledge of various mathematical topics (e.g., number and operations, algebra) and domains (e.g., knowledge of content; knowledge of students and content). The constructs indicated by factor analysis form psychometrically acceptable scales.

In the past two decades, teachers' knowledge of mathematics has become an object of concern. New theoretical and empirical insights into the work of teaching (e.g., Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987) have spurred greater attention to the role played by such knowledge in teacher education and the quality of teaching itself (e.g., NCTAF, 1996). Other studies have documented the mean and variation in teachers' knowledge of mathematics for teaching (e.g., Ball 1990; Ma 1999). Results of these efforts have been reflected in teaching standards published by Interstate New Teacher Assessment and Support Consortium (INTASC), the National Board for Professional Teaching Standards, as well as by many other states, localities, and professional teaching organizations (e.g., NCTM). Concerns that teachers possess necessary knowledge and skills for teaching mathematics have also led to the development and use of teacher licensing exams, such as PRAXIS, an assessment developed by the Educational Testing Service and now administered in 38 states. Other states and testing firms have developed and administer similar assessments.

Given the development of such standards and assessments, one might conjecture that there is substantial agreement around the knowledge needed for teaching children mathematics. However, a closer look at released items from the elementary mathematics portion of these teacher licensure exams suggests lack of agreement over what teachers actually need to know to teach this subject. Some exams assess individuals' capability in solving middle-school level mathematics problems (California's CBEST; PRAXIS), others the ability to construct mathematical questions and tasks for students (Texas' EXCET), and still others the ability to understand and apply particular mathematics content areas to teaching (Massachusetts' MTEL). This implicit disagreement over the content and nature of teachers' professional knowledge of mathematics can be traced backwards, through the theoretical and empirical literature on teaching knowledge, where different authors propose divergent elements and organizations for such knowledge. It can also be traced forward through current debates about the mathematics teachers need to know to teach. Some argue, for instance, that individual capability in general mathematics is the most important qualification for teaching this subject (U.S. Department of Education, 2002). Others take the view that general mathematical ability must be complemented by additional professional knowledge, such as knowledge of student thinking about content, or mathematical tasks specific to the work of teaching. To date, however, little empirical data has been publicly available to help judge the validity of either claim.

We seek to shed light on this debate by analyzing data collected in the service of constructing an assessment of teachers' content knowledge for teaching mathematics. To construct this assessment, we used elements from existing theories about teacher knowledge (e.g., Ball & Bass, 2001; Grossman, 1990; Shulman et al, 1987) to write a set of survey-based teaching problems thought to represent various components of the knowledge of mathematics needed for teaching. We then factor analyzed teachers' responses to this item set to determine the structure of the knowledge we tried to represent. The principal question guiding our work is: Is there one construct which can be called "mathematics knowledge for teaching" and which explains patterns of teachers' responses, or do these items represent multiple constructs, and thus several distinct mathematical competencies held by practicing elementary mathematics teachers? A second question is: Given the structure of teachers' mathematical knowledge for teaching, can we construct scales which measure such knowledge reliably?

In this paper we describe this effort and its results. We begin with an overview of the original literature about content knowledge for teaching. Next we shift to a discussion of our own efforts to write items that represent such knowledge, with an emphasis on the potential constructs that might emerge from the items. Finally, we describe initial results from a field-test of these items, including factor analyses and attempts to scale these items for use in statistical work.

Literature Review

In the mid-1980s, Lee Shulman and his colleagues introduced the notion of "pedagogical content knowledge" to refer to the special nature of the subject matter knowledge required for teaching (Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). Conceived as complementary to general pedagogical knowledge and general knowledge of subject matter, the concept of pedagogical content knowledge was thought to include familiarity with topics children find interesting or difficult, the representations most useful for teaching a specific content idea, and learners' typical errors and misconceptions. Labeling this as "pedagogical content knowledge" not only underscored the importance of understanding subject matter in teaching, but it also suggested that personal knowledge of the subject — that is, what an educated adult would know of the subject — was insufficient for teaching that subject. This distinction represented an important contribution to the puzzles about qualities and resources needed for effective teaching.

This line of research focused on subject matter knowledge, and conceived of such knowledge as particular, rooted in the details of school subject matter and of what is involved in helping others understand it. Working in depth within different subject areas, scholars probed the nature of the content knowledge entailed by teaching; comparisons across fields were also generative. Grossman (1990), for example, articulated how teachers' orientations to literature shaped the ways in which they approached particular texts with their students. And Wilson and Wineburg (1987) illuminated ways in which social studies teachers' disciplinary backgrounds shaped the ways in which they represented historical knowledge for high school students. In mathematics, scholars showed that what teachers would need to understand about fractions, place value, or slope, for instance, would be substantially different from what would suffice for other adults (Ball, 1988, 1990, 1991; Borko, Eisenhart, et al., 1990; Leinhardt & Smith, 1985).

Despite this wealth of research, we argue the actual mathematical content that teachers must know to teach has yet to be precisely mapped. Most of the foundational work in this area has relied principally on single-teacher case studies, expert-novice comparisons, cross-national comparisons, and studies of new teachers. While such methods have been critical in beginning to articulate the content of subject matter knowledge for teaching, these methods lack the power to propose and test hypotheses regarding the organization, composition and characteristics of content knowledge for teaching.

Researchers have, however, conjectured about the potential organization of such knowledge, and these conjectures prove useful starting points for this investigation.

Shulman (1986) originally proposed three categories of subject matter knowledge for teaching. His first category, *content knowledge*, "refers to the amount and organization of knowledge per se in the mind of teachers" (p.9). Content knowledge, according to Shulman, included both facts and concepts in a domain, but also why facts and concepts are true, and how knowledge is generated and structured in the discipline (Bruner, 1960; Schwab, 1961/1974). The second category advanced by Shulman and his colleagues (Shulman, 1986; Wilson, Shulman, & Richert, 1987), *pedagogical content knowledge*, "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9). This category of subject matter knowledge for teaching has become of central interest to researchers and teacher educators alike. Included here are representations of specific content ideas, as well as an understanding of what makes the learning of a specific topic difficult or easy for students. Shulman's third category of subject matter knowledge for teaching, *curriculum knowledge*, involved awareness of how topics are arranged both within a school year and over time and ways of using curriculum resources, such as textbooks, to organize a program of study for students.

Shulman's theory of teacher knowledge listed also general pedagogical knowledge (classroom management techniques and strategies), knowledge of learners and their characteristics, knowledge of educational contexts (e.g., school board politics, communities), and knowledge of educational ends, purposes, and values.

Leinhardt & Smith (1985) proposed a different organization of teacher knowledge in their study of expert-novice differences in mathematics teaching. Working from a psychological/cognitive perspective, they identified two aspects of knowledge for teaching: lesson structure knowledge – which includes planning and running a lesson smoothly and providing clear explanations – and subject matter knowledge. They include in the latter “concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number systems being drawn upon, the understanding of classes of student errors, and curriculum presentation” (p.247).

Other ways of dividing up the terrain have been advanced as well. Grossman (1990) reorganized Shulman and colleagues’ categories into four, and extended them slightly: subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge (knowledge of students’ understanding, curriculum knowledge, knowledge of instructional strategies) and knowledge of context. Ball (1990) described differences between teachers’ ability to execute an operation (division by a fraction) and their ability to represent that operation accurately for students, demarcating clearly two dimensions in teachers’ content knowledge –the ability to calculate a division involving fractions, and the kind of understanding of that operation needed for teaching. And, based on analyses of classroom lessons, Ball proposed a distinction between knowledge of mathematics and knowledge about mathematics, corresponding roughly to knowledge of concepts, ideas, and procedures and how they work, on one hand, and knowledge about “doing mathematics” — for example, how one decides that a claim is true, a solution complete, or a representation accurate. In more recent work, Ma (1999) used comparisons of U.S. and Chinese elementary teachers to describe “profound understanding of fundamental mathematics” as instantiated in the connectedness, multiple perspectives, basic (fundamental) ideas, and longitudinal coherence which occur during their teaching.

By posing these potential maps of content knowledge for teaching, researchers have contributed to the development of theory about the content knowledge needed for teaching. One contribution has been to refocus the attention of the educational research community on the centrality of subject matter and subject matter knowledge in teaching. A second was to draw attention back to disciplines and their structures as a basis for theorizing about what teachers should know. A third has been to focus attention on what expert teachers know about content and how they use or report using this knowledge of subject matter in their teaching.

While this work has contributed to the field, much remains to be done. For example, there is still much to be understood about the organization and structure of subject matter knowledge within different disciplines and what these structures suggest for teaching. Little is known yet about whether and how content knowledge for teaching relates to the knowledge of the content held by other professionals or by ordinary educated adults. And to date, scholars have not attempted to measure teachers' knowledge for teaching in a rigorous manner, and thus cannot track its development or contribution to student achievement.

Method

To learn more about these issues, we began in 2001 to write, and later pilot, large numbers of multiple-choice items intended to represent the mathematical knowledge used in teaching elementary mathematics. Item-writing served several purposes: at the most practical level, we hoped to develop measures by which we could gauge growth in teachers' content knowledge for teaching, and learn more about how such knowledge contributes to student achievement. Item-writing also served as another way to explore the nature and composition of subject matter knowledge for teaching. During the process of examining materials and student work, writing and refining items, and thinking about what items represented, we sharpened and defined our ideas about the mathematical knowledge and skill needed for teaching mathematics. Finally, piloting these items served an intermediate purpose, allowing us to use factor analyses and scaling techniques to learn about the organization and characteristics of mathematical knowledge for teaching. Before describing the results of our analyses and efforts to build scales, we recount the process by which we developed survey items, and describe the possible ways these items might be categorized.

Developing survey items. Our approach to studying content knowledge for teaching was grounded in a theory of instruction, taking as a starting point the work of enacting high quality instruction (Ball & Bass, 2000; Ball & Cohen, 1999; Cohen & Ball, 2000). From that perspective, we asked, "What mathematical knowledge is needed help students learn mathematics?" Our interest was in identifying what and how subject matter knowledge is required for the work of teaching. Using this theoretical perspective as well as the research base, analyses of curriculum materials, examples of student work, and personal experience, researchers at [name of project] developed 138 mathematics items in the spring of 2001.

Table 1 about here

Researchers wrote mathematics items to reflect categories shown in Table 1. Two of the mathematical content areas – number concepts and operations – were selected because they comprise a significant portion of the K-6 curriculum and because important and useful work existed on the teaching and learning of these topics. Patterns, functions, and algebra was chosen because it represents a newer strand of the K-6 curriculum, and thus allows insight into what and how teachers know about this topic now, and perhaps how knowledge increases over time, as better curriculum and professional development becomes available, and as teachers gain experience with teaching this topic. Initial item-writing efforts also focused on two kinds of teacher knowledge: knowledge of content itself and combined knowledge of students and content. By demarcating the domains in this way, writers intended to reflect elements contained in Shulman and others' typologies of content knowledge for teaching. Crossing the three content areas with the two domains of teacher knowledge yielded six cells. However, the lack of research and other specific resources on students' learning of patterns, functions, and algebra led us to conclude we could not develop items in this cell during these initial item-writing efforts (see Table 1).

The constructs, or underlying organizing principles, indicated by factor analyses with these items might reflect the five existing domains exactly. Yet a post-hoc analysis of the items revealed other potential hypotheses about the organizational structure. To start, there may be only one construct that explains the patterns in teachers' responses to items in all five cells. If this were so, we might conjecture that this single construct could be described as "general mathematical ability," and might conclude that there is little need to specifically identify specialized knowledge for teaching, or that this specialized knowledge is so strongly related to the knowledge held by other educated adults so as to be functionally the equivalent, at least for measurement purposes. At the other end of the spectrum, however, we might find that items are differentiated at a much finer grain size than that reflected in Table 1. For instance, teachers' knowledge might be differentiated at the level of particular topics in the elementary curriculum – e.g.. whole numbers, fractions, decimals, operations (e.g., addition) with whole numbers, etc. If this were the case, we might conjecture that teachers have highly particularized knowledge of the material they teach, and we would study these knowledge clusters in more depth.

Mathematical content areas are not the only potential organization of items. These items were situated in yet another possible categorization system, what we would call tasks of teaching. This way of categorizing items is based on the idea that teachers' mathematical knowledge is used in the course of different sorts of tasks — choosing representations, explaining, interpreting student responses, assessing student understanding, analyzing student difficulties, evaluating the correctness and adequacy of curriculum materials. These are tasks teachers might face in any subject matter and they provide another potential organization of teachers' content knowledge for teaching mathematics.

Finally, items may differentiate themselves within the cells shown in Table 1. For instance, some items appear to require respondents to draw on common knowledge of content (CKC) – for instance, items that ask teachers to find the decimal halfway between 1.1 and 1.11, or to find the eighty-third shape in a sequence. As Shulman and others point out, such mathematics knowledge is used in the course of teaching, as teachers must compute, make correct mathematical statements, and solve problems. Other items, however, appear to be based on the particular ways mathematics arises in elementary classrooms, or what we call specialized knowledge of content (SKC), including building or examining alternative representations, providing explanations, and evaluating unconventional student methods. One way to illustrate this distinction is by imagining how someone who has not taught children but who is otherwise knowledgeable in mathematics might interpret and respond to these items. This test population would not find the items that tap ordinary subject matter knowledge difficult. By contrast, however, these mathematics experts might be surprised, slowed, or even halted by the mathematics-as-used-in-teaching-items; they would not have had access to or experience with opportunities to see, learn about, unpack and understand mathematics as it is used at the elementary level.

Two of the mathematics items included in Appendix A illustrate this distinction. In the first item, about powers of ten, teachers must draw on their knowledge of properties of numbers – in this case, place value as represented within exponential notation—to answer the problem. This content knowledge is used in teaching; students learn about exponential notation in the middle to late elementary grades, and thus it follows that teachers must have adequate knowledge to develop this topic. However, many adults, and certainly all mathematicians, would know enough to answer this item correctly. The next item illustrates a special kind of content knowledge, one that arises through the teaching of content to young children. In the second item, teachers must inspect three different solutions to the same two-digit multiplication problem – 35 x 25 – and assess whether approaches used for each solution would generalize to all whole number multiplication. To respond in such situations, teachers must draw on their mathematics knowledge – inspecting the solution to understand what was done at each step, then gauging whether the method makes sense and would work in all cases. Analyzing procedures and justifying their validity is a mathematical process. However, doing it in this way and in this context (i.e., appraising different student solutions to a computation problem) is a task that arises regularly in teaching, and not necessarily in other arenas. Hence, it is a specific form of and context for mathematical reasoning in which teachers must engage and it appears to draw on a specialized form of mathematical knowledge, one that makes it possible to analyze and make sense of a range of methods and approaches to a particular computation.

The knowledge of students and content (KSC) category also contained subtle distinctions. As we (and mathematicians associated with this project) reviewed items, we saw that some such items required knowledge of students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems. Teachers may need to know, for instance, what errors students make as they learn about the place value system, or strategies students might use to remember the answer to 8 x 9. In other cases, knowledge of students and content items might draw on student thinking and/or mathematics content knowledge. For instance, teachers may use both types of knowledge in order to interpret student statements about the commutative property, analyzing what students have said about this topic to assess understanding and depth of knowledge.

The third and fourth problems in Appendix A illustrate distinctions within our knowledge of students and content (KSC) category. The third item asks teachers to consider which of three lists of decimal numbers would be best to assign to discriminate students' understanding of and skill with ordering decimal numbers, or whether any of the three lists would be equally useful for teachers to use for this purpose. Two of the three lists would allow students to respond correctly without paying any attention to the decimal point at all. In interviews, we have seen that many people who have never taught this topic, including mathematicians, see no difference among the three lists; teachers with knowledge of decimals for teaching are more likely to see the differences immediately.

Thus knowledge of students, and typical mistakes on this item, is necessary to correctly answer this problem. The next item, on buggy algorithms, requires either knowledge of typical student mistakes or the ability to perform a detailed mathematical analysis to arrive at a correct answer.

Data collection and analysis. Items were piloted in California's Mathematics Professional Development Institutes. These institutes were publicly funded, large-scale efforts to boost California teachers' knowledge of subject matter in mathematics. The institutes had over 40 sites, cost roughly $65 million, and served 23,000 K-12 teachers in the first three years of the program. Piloting took place with only elementary teachers enrolled in Number and Operations institutes. At a typical Number and Operations institute site, teachers were paid up to $1500 to attend summer sessions ranging from one to three weeks in length. Academic mathematicians and mathematics educators were the instructors in these institutes; the content was mathematics — number and operations. While MPDI sites were selected on the basis of their willingness to partner with low-performing/high poverty districts and schools, teachers were not identified for recruitment on the basis of pre-existing mathematical knowledge or other characteristics (Madfes, Montell & Rosen, 2002). As a condition of funding, each institute was required to administer an evaluation designed to gauge growth in teacher content knowledge. By supplying these measures, (name of authors' project) and officials at California's MPDIs formed a mutually beneficial partnership, allowing both the piloting of items and, potentially, an evaluation of the institutes' effectiveness. Items and forms, however, were not written or constructed to align with any particular MPDI, since content varied across the 21 institutes included in this analysis, and since we wanted to design measures that could be appropriately used beyond the MPDI setting.

By combining the summer pre/post assessments given to teachers, we were able to obtain enough responses to each of three pilot forms: 640 cases for form A, 535 for form B, and 377 cases for form C, to conduct statistical analyses. Each form was constructed such that roughly 7 stems and 11-15 items represented each cell in Table 1. Within each cell, three “linking items” were constant across all three forms; these linking items allow for form equating in the evaluation portion of this project, but also allow us to test and confirm hypotheses about particular items across the three different forms. The remaining items within a cell differed between forms, but still followed the general themes and topics for items outlined above. Thus factor analyses done on each of the three forms could return consistent results broadly (e.g., finding the same number of factors, interpreting factors in the same way) and for a small number of linking items. To perform factor analyses, we used a program written to accommodate items linked by common stems or scenarios such as item 2 in Appendix A (ORDFAC; [author] 2002). To learn more about other item characteristics, we used BILOG (Mislevy & Bock, 1997), a program which enables item response theory analyses (Hambleton, Swaminathan, Rogers 1991).

Results

In this analysis, we answer two questions: how teachers’ mathematical knowledge for teaching is organized within mathematics; and whether we can, with these items, reliably measure teachers’ mathematical knowledge for teaching. The results we present here draw on the descriptions of data analysis presented in [author] (2002); readers who wish a more thorough and technical version of the factor analysis story should refer to this manuscript.

How might content knowledge for teaching be organized conceptually? As we described above, the items used in the MPDIs have multiple potential organizations. By putting all items on each form into ORDFAC, we can determine which items relate to the same underlying constructs, how many such constructs exist, and with what certainty we can identify this structure.

This task is complicated by the limitations of the analytic method of item factor analysis. Item factor analysis identifies patterns of association between items for a particular sample answering a particular survey instrument. A pattern of association is a necessary but not sufficient condition for identification of a unidimensional construct. Items measuring conceptually different constructs can also show a pattern of association in item factor analysis because of a strong correlation between the underlying constructs in that sample. Another sample, in which the same constructs do not exhibit as strong a correlation, will often show a different pattern of association, differentiating the two conceptually distinct constructs. As discussed below, this phenomenon was exhibited in two of our forms, where the patterns, functions, and algebra content items loaded on the same factor as the knowledge of students and content items in number concepts and operations. A related question is, assuming a number of conceptually distinct but related constructs for a set of items, the extent to which a single "general" factor can account for the covariation between items compared to the amount of covariation accounted by specific factors. Addressing this issue provides insight to the meaning that might be attached to the use of a simple total score for our instruments. In order to address these concerns, we chose to examine the data using three types of analysis: a) exploratory factor analyses of the three forms; b) factor analyses with patterns, functions, and algebra items removed, to seek additional clarity of results; c) bi-factor analyses, to further assess the issue of multidimensionality and to resolve questions regarding knowledge of students and content items.

Exploratory factor analysis of all items on form A suggested that there were three underlying dimensions: a) knowledge of content in number concepts and operations; b) knowledge of content in patterns, functions, and algebra; and c) knowledge of students and content in number concepts and operations. This is illustrated in Table 2, which presents Promax rotated factor loadings for all items. Note that all of the knowledge of number concepts and operations items, with one exception, load strongly on the first factor and all the knowledge of patterns, functions and algebra items load on the third factor. The situation for the knowledge of students and content items is more complicated. Most of these items (9 out of 14) load primarily on the second factor, but a significant minority load primarily on the first factor. This suggests either knowledge of content or knowledge of students and content might alternatively be critical for answering these types of items correctly. However, inspection of the wording of each item failed to reveal any noticeable difference between the items loading on the first factor and items loading on the second factor.

We also ran factor analyses on all items for forms B and C. The results of these analyses were consistent with form A with respect to the knowledge of number concepts and operations items loading almost exclusively on the first factor and the knowledge of patterns, functions, and algebra items loading exclusively on the third factor. However, the results for knowledge of students and content items differed across the three forms. In forms B and C these items loaded most often on the first (content) and third (algebra) factors – the second factor had only a few items with strong loadings on both forms.

Conceptually, there was little reason to believe that the student thinking and patterns functions and algebra items should be combined to form a scale. Instead, it is likely that these two constructs are correlated in the form B and C samples, as described above. This combined with the results of the analysis of form A suggested that the presence of the patterns, functions, and algebra items might be obscuring the relationships between the student thinking items. Therefore we omitted these items from subsequent analyses and focused on whether knowledge of content and knowledge of students and content are distinguishable factors.

Results suggest that we can make such a differentiation. To start, we fit exploratory factor models of increasing complexity (number of factors) to these items. The results of these successive fits for each form are presented in Table 3. Schilling and Bock (2003) recommend that a model of increasing complexity only be accepted if the chi-square statistic for a model is two times the difference in the degrees of freedom between the two models. This heuristic is also employed in the Akaike Information Criterion (AIC) (Agresti, 1990) were a low value indicates better fit. By both criterion, a two factor model provided the best fit for form A, while three factor models provided the best fit for forms B and C. Table 4 presents the loadings for the two factor models for all three forms while Table 5 presents the loadings of the three factor models for form C. Examination of the factor loadings for the two and three factor models for form B revealed the two models to be essentially the same, with the exception that three of the items for the knowledge of students and content in operations items comprised a third factor for the latter model. The two and three factor models for form C differed in that none of the knowledge of students and content in number concepts items had substantial loadings on the second factor for the two factor model, but four of the eight items loaded on a third factor for the three factor model.

Taken together, these exploratory analyses, in the main, suggested at least three dimensions across all the items reflecting the following constructs:

- Knowledge of content (KC) in elementary number and operations
- Knowledge of students and content (KSC) in elementary number and operations
- Knowledge of content (KC) in patterns, functions, and algebra

Although results differ across forms in the area of knowledge of students and content, these results correspond to the categorization system shown in Table 1, assuming the combining of number and operations.

After these analyses suggested this general shape to our data, we ran a five-factor bi-factor model. This model specifies the number of factors (five), and allows each item to load in two places: on a general factor which explains teachers' responses to all items, and on a specific factor representing its place in the categorization scheme described in Table 1 – although because exploratory factor analysis soundly showed no difference between number and operations content items, we assigned both sets of items to only one factor. We tested this bi-factor model for three reasons. First, results will assist us in determining to what extent a general factor vs. specific factors explain patterns in teachers' responses to these items. Second, results will allow us to better assess our hypothesis that there is a difference between common and specialized knowledge of content. Here we might expect that items that tap common knowledge should load on the general factor, items that tap specialized knowledge would load on a specific factor. Finally, results will allow us to better understand the multidimensionality within the knowledge of students and content items.

Results from this bi-factor analysis are informative. First, the general factor explained between 67-77% of the overall variation in teachers' responses to items on each of the three forms (see Table 6). This factor explained variation in a substantial number of content knowledge items, suggesting that this factor can be interpreted as common knowledge of content (CKC), and suggesting an influence of general grasp of mathematics in patterning teachers' responses to items. However, multi-dimensionality is also apparent here, as the factors specifically describing knowledge of students and content (KSC) and knowledge of content (KC) in patterns, functions, and algebra typically account for between 25-47% of the communality in items written to represent these areas. Further, the specialized content knowledge (SKC) factor explained between 5-30% of the communality of items written to represent knowledge of content in number and operations.

Similar to results from the exploratory analysis, some knowledge of students and content items continued to load on the common knowledge of content (CKC) factor, others loaded on their own factor, and many loaded on both. There were no firm patterns among items in how they loaded; both the CKC and KSC factors included items that referenced student errors, common strategies, similar subject matter content, and a range of item difficulties. Whatever the cause of these loading patterns, it makes sense to think that mathematical content knowledge and knowledge of student and mathematics should be intertwined, for it is difficult to imagine teachers having strong knowledge of students' learning without some basic knowledge of the mathematics they study.

Multi-dimensionality is also apparent in the items written to represent knowledge of content in elementary number and operations. To a large extent, items representing common knowledge of content (CKC) tended to appear on the general factor, suggesting again that this factor represents overall mathematical ability. However, variation in teachers' responses to items written to represent specialized knowledge of content (SKC) was much more likely to be explained, at least in part, by the "specific" content knowledge factor. Finding this factor supports the conjecture that there is content knowledge used in teaching that is specific to key tasks teachers must engage. Inspecting the items which comprise this factor further support this hypothesis, and suggest some basic outlines to this knowledge. These items included those that engage teachers in:

- Analyzing alternative algorithms or procedures
- Showing or representing numbers (e.g., 10.05) or operations (e.g., 1/2 x 2/3) using manipulatives
- Providing explanations for common mathematical rules (e.g., why any number can be divided by 4 if the last two digits are divisible by 4)

These correspond closely to our initial ideas about the constituent parts of specialized knowledge of content, with one exception: items which asked teachers to match fractions number sentences to stories (e.g., represent 1 1/4 ÷ 1/2 with a story) appeared on the general factor. Nevertheless, finding this specific factor supports the idea of specialized knowledge of content.

Evidence that supports the existence of the specialized content knowledge for teaching is important. From a measurement standpoint, these results suggest that common and specialized mathematical ability are related, yet are not completely equivalent; the possibility exists that individuals might have well-developed common knowledge, yet lack the specific kinds of knowledge needed to teach. It also suggests that individuals might develop the specialized knowledge for teaching mathematics – perhaps from teacher preparation, professional development, working with students or curriculum materials – without having otherwise expert knowledge of mathematical content. This finding has implications for theory, policy, teacher preparation, and measurement. We discuss some of these below.

Apart from the major structure of the data, there are several things to note about these findings. First, these items did not organize themselves around generic tasks of teaching (e.g., evaluating curriculum materials, interpreting students' work). Instead, these results suggest the organization of teachers' knowledge is at least somewhat content-specific. Yet these constructs were not highly particular, either: instead of finding highly specific factors that represent either content (e.g., fractions, whole number computation) or very specific tasks of teaching mathematics (e.g., representing numbers and operations, analyzing student errors), we found broader groupings of items. Finally, the items that appear on all three forms tended to perform consistently across those forms in our factor analyses, with only minor exceptions (see Author, 2002).

Overall, results from these factor analyses suggest that teachers' content knowledge for teaching is at least somewhat domain-specific, and that scholars who have hypothesized about the categories around which teacher knowledge might organize are at least partially correct. Subject matter content does play a role; so do the different ways mathematical knowledge is used in classrooms. Including additional content areas (geometry, data and statistics) and a fuller array of knowledge of students and content items (e.g., in algebra, geometry) would allow for further testing of this finding. In the meantime, we consider these findings' implication for constructing measures of teacher knowledge.

Can we measure content knowledge for teaching? Given the results from the factor analysis, can we construct reliable measures that accurately represent teachers' ability in these areas? This was a major goal of our work, for these measures are needed to gauge the effectiveness of various forms of professional development and teacher learning, and to estimate the contribution of teacher knowledge to student achievement.

We used BILOG to fit initial item response theory (IRT) models to the data (Hambleton, Swaminathan, & Rogers 1991). We present results for scales for a) each cell in Table 1, b) for combined number concepts/operations knowledge of content and knowledge of students and content scales, and c) for an overall measure of mathematical knowledge for teaching. Table 7 provides descriptive statistics for these scales on each of the three forms – coefficient alpha for a classical test theory measure of reliability, IRT reliabilities computed using BILOG, and points of maximum test information. The reliabilities for patterns, functions, and algebra scales, as well as scales that combine number and operations items within each domain, are good to excellent, ranging from 0.71 to 0.84. However, the points of maximum information reveal how each scale could be improved. The lowest reliability of 0.71 occurs for the knowledge of students and content scale on form A where the maximum information was 1.9 standard deviations below the population mean. In contrast, the number concepts/operations content knowledge scale for form A has point of maximum information at 0.36 standard deviations below the population mean. This means that this scale is better targeted to the skill level of the population, hence the scale has a higher reliability – 0.81.

There are several things to note about these efforts to build measures that reflect individuals' content knowledge for teaching mathematics. First, measures representing teachers' knowledge of content had higher reliability than those composed of items meant to measure familiarity with students and content. Second, for most measures, the test provided the most information (test information curve maximum) at abilities below the average teacher; that is, items were, on average, too easy, yielding the best measurement (lowest standard errors) for teachers who fell between .5 and 2.0 standard deviations below average. This trend was most pronounced in the knowledge of students and content measures. Third, there remain some significant problems with multidimensionality with these items, particularly in the areas of knowledge of students and content and, for those who choose to use this construct, the specialized knowledge of content. For more on potential solutions to this problem, see [author], (2002).

Finally, any appraisal of the utility of a particular measure must include an examination of the relationship between individuals' performance on the instrument and those individuals' real skill or ability – i.e., validity. For these measures, a best-case investigation of validity would include comparing teachers' measure score with an assessment of their use of mathematics content in actual classroom teaching. This work is currently under way at (name of project). Less convincing, although more often done in the field of test construction, is cognitive tracing interviews, in which individuals talk through their thinking and answers to particular items. If individuals' thinking does not reflect their answers, problems of validity are likely. An analysis conducted with similar items to these suggested that for knowledge of content items, teachers' answers did in fact represent their underlying reasoning process; results were not so sanguine for student thinking items, where more problems pertained (see author, 2002). This suggests the more varied factor analysis results and lower reliabilities for this second set of items may be grounded in problems with measurement in this domain.

Conclusion

By developing measures of teacher knowledge for teaching mathematics, we hope to contribute to a number of ongoing efforts in educational research to answer policy-relevant questions: identifying the effect of teacher knowledge on student achievement, explaining how teacher knowledge develops (via experience, professional training, professional development), and answering other key policy questions (e.g., the effects of certification on teacher knowledge). However, we believe developing such measures can also contribute to a renewal of interest in the theoretical aspects of professional knowledge for teaching. By allowing insight into how knowledge is held by teachers, how that knowledge relates to common subject-matter knowledge, and perhaps even (through open-ended interviews) how teachers, non-teachers, and subject matter experts deploy knowledge.

We see the analyses above as a first step in the measures development process. The dataset was less than ideal, since teachers were non-randomly sampled and MPDI pre- and post-tests were combined for this analysis. Because different subjects answered different forms on the pretest and posttest, this did not present significant problems for our use of IRT, other than perhaps producing a non-normal distribution of ability in the sample for a particular form. Fortunately, IRT models are generally robust to non-normal distributions of ability (Bock and Aitken, 1981) We also measured typical, rather than expert, teachers, and this may further constrain our results: if typical teachers do not have or have less specialized knowledge for teaching mathematics, we bias our results toward a null finding for this hypothesis. And these findings should also be replicated, both through additional studies similar to the one reported here, but also through the use of multiple methods, including interviews and observations of classroom instruction.

However, our analyses suggest some tentative results can be reported now. First, repeated analyses across three different forms found evidence for multi-dimensionality in these measures, suggesting that teachers' knowledge of mathematics for teaching is at least partly domain-specific, rather than simply related to a general factor such as overall intelligence, mathematical or teaching ability. While results from the bi-factor analysis suggest such a general factor does operate, additional communality is explained by specific dimensions; this supports Shulman and others' claims that knowledge for teaching consists of both general knowledge of content and more specific domains.

The domains identified in the factor analyses are themselves interesting. Our data suggest that in addition to a general factor, specific factors represent knowledge of content in number and operations, knowledge of students and content in number and operations, and the relatively newer area (for elementary school) of knowledge of content in patterns, functions, and algebra. The data also suggest a specialized knowledge of content (SKC) measure made up of several types of items: representing numbers and operations, analyzing unusual procedures or algorithms, and providing explanations for rules. Writing items which represent more content areas, more specialized tasks (e.g., using mathematical definitions in teaching) and possibly more domains (e.g., knowledge of teaching and content), will allow us to assess the extent to which content and task continue to play a role in defining domains of teacher thinking.

Our findings suggest lessons for theory, policy, and measurement. First, these results provide evidence for the conjecture that content knowledge for teaching mathematics consists of more than the knowledge of mathematics held by any well-educated adult. While it appears that such knowledge of mathematics is an important component of the knowledge needed for teaching, there may be more mathematical depth to teaching elementary school, in other words, than simply the content of a third, fifth, or even eighth grade textbook. We cannot definitively say that teachers must know these specific areas in order to help students learn – such a statement must wait for the results of analyses that compare the effects of these different kinds of knowledge on growth in classrooms. But it does hint that rather than focusing simply on how much mathematics an individual knows, as has historically been the case (see Shulman, 1986), we must also ask how that knowledge is held and used by the individual – whether they can use their mathematical knowledge to generate representations, interpret student work, or analyze student mistakes. It also suggests the utility of continuing to identify the content, so to speak, of our specialized knowledge of content category, and thus extending our notions of the knowledge needed to teach.

If our results hold, these findings also bear on current policy debates regarding the recruitment and preparation of teachers. Strong knowledge of basic mathematical content does matter; however, policy-makers must take seriously the idea that additional capabilities may be layered atop that foundation. Until we can replicate these results, we cannot definitively say teachers should learn this information in pre-service or in-service preparation. Yet finding evidence for these multiple dimensions lends support for a curriculum that goes into depth, and that is highly specific to the work of teaching. Teachers may need to know why mathematical statements are true, how to represent mathematical ideas in multiple ways, what is involved in an appropriate definition of a term or a concept, and methods for appraising and evaluating mathematical methods, representations, or solutions. By helping teachers develop knowledge of mathematics that goes beyond the sort of understanding needed for everyday non-professional functioning, faculty and professional developers may assist teachers in preparing for the tasks they will encounter on the job.

From a policy perspective, our research suggests supporting professional development and teacher preparation programs which enable this kind of learning. However, it also carries a lesson for those in business of constructing teacher licensure exams, at least at the elementary level; reviews of several currently used exams suggest the majority of problems simply ask teachers to compute, rather than to use knowledge more classroom-authentic ways. If we find that the more specific kinds of expertise identified here affect student achievement -- or even if we simply decide, based on normative arguments, that teachers should possess this knowledge – licensure exams should reflect this emphasis.

From a measurement perspective, these results suggest constructing separate scales to represent knowledge for teaching mathematics. This is an important point for researchers, who aim to devise measures that are sensitive to differences in individuals' unique combinations of knowledge and skills in order to explore relationship between such measures and others like student achievement. The presence of multi-dimensionality also changes way we might model teacher development and contribution to student achievement; rather than using one catch-all variable, we can contrast impact of growth in various domains on student achievement, and predict the impact of growth in various domains from various "treatments," such as the effect of the first years of teaching on knowledge of student strategies, mistakes, and methods.

Table 1: Mathematics content areas and domains

	Domains	
	Knowledge of content	Knowledge of students and content
Number concepts		
Operations		
Patterns, functions, algebra		

NOTE: Shaded area represents construct for which no items were developed

Table 2: Promax rotated factor loadings, Form A

Item	F1	F2	F3
NCKC1	**0.512**	0.138	0.063
NCKC2	**0.473**	0.113	-0.059
NCKC3	**0.260**	-0.132	0.165
NCKC4	**0.219**	0.062	-0.180
NCKC5	**0.444**	0.158	-0.133
NCKC6	**0.228**	0.097	0.086
NCKC7	0.139	-0.039	**0.295**
OPKC1	**0.732**	0.068	-0.203
OPKC2	**0.246**	0.084	0.136
OPKC3	**0.637**	-0.210	0.101
OPKC4	**0.643**	0.042	0.036
OPKC5	**0.704**	-0.292	-0.023
OPKC6	**0.511**	-0.143	-0.052
PFAKC1	-0.290	-0.111	**0.773**
PFAKC2	0.259	-0.038	**0.403**
PFAKC3	0.015	-0.016	**0.675**
PFAKC4	0.156	0.175	**0.314**
PFACK5	0.337	-0.047	**0.419**
PFACK6	0.039	-0.113	**0.639**

NCSC1	0.061	**0.275**	0.121
NCSC2	**0.263**	**0.327**	0.158
NCSC3	**0.311**	0.149	-0.021
NCSC4	0.002	**0.365**	0.010
NCSC5	-0.109	**0.248**	0.212
NCSC6	0.180	**0.386**	0.047
NCSC7	**0.352**	-0.041	0.019
OPSC1	-0.055	**0.946**	-0.098
OPSC2	**0.466**	0.181	-0.058
OPSC3	0.018	**0.249**	0.130
OPSC4	**0.493**	-0.022	0.017
OPSC5	-0.125	**0.699**	-0.124
OPSC6	-0.012	**0.417**	-0.022
OPSC7	-0.015	0.149	0.151

Table 3: Exploratory Factor Analyses Number Concepts/Operations

Form A Exploratory	chi-sq	df	AIC
1 FACTOR			25345
2 FACTOR	96.96	26	25300
3 FACTOR	44.82	25	25305
4 FACTOR	43.34	24	25310
Form B Exploratory			
1 FACTOR			22506
2 FACTOR	72.76	28	22489
3 FACTOR	66.20	27	22477
4 FACTOR	47.76	26	22481
Form C Exploratory			
1 FACTOR			16219
2 FACTOR	111.98	29	16165
3 FACTOR	69.46	28	16152
4 FACTOR	37.56	27	16168

Table 4: Number Concepts & Operations Promax Rotated Factor Loadings - Two Factor Models

	Form A		Form B		Form C	
Item	F1	F2	F1	F2	F1	F2
NCKC1	**0.546**	0.159	0.253	**0.491**	**0.641**	-0.067
NCKC2	**0.456**	0.059	0.192	**0.342**	**0.473**	0.026
NCKC3	**0.373**	-0.126	0.174	**0.238**	**0.222**	0.027
NCKC4	0.126	0.016	**0.408**	-0.012	0.193	**0.267**
NCKC5	**0.357**	0.130	**0.353**	0.191	**0.616**	-0.118
NCKC6	**0.273**	0.110	**0.591**	0.269	**0.551**	-0.181
NCKC7	**0.328**	0.004	**0.327**	**0.457**	**0.697**	-0.090
NCKC8			-0.126	**0.517**		
NCKC9			**0.594**	0.068		
NCKC10			**0.454**	0.247		
OPKC1	**0.606**	0.031	**0.844**	-0.084	**0.691**	-0.005
OPKC2	**0.338**	0.090	**0.395**	0.098	**0.411**	0.012
OPKC3	**0.761**	-0.250	**0.552**	-0.140	**0.415**	0.043
OPKC4	**0.681**	0.027	0.291	**0.402**	**0.234**	**0.326**
OPKC5	**0.699**	-0.304	**0.321**	0.230	**0.386**	0.101
OPKC6	**0.503**	-0.176			**0.659**	0.004
OPKC7					**0.492**	-0.071
OPKC8					**0.295**	**0.366**
NCSC1	0.164	**0.264**	0.052	**0.217**	0.128	0.188
NCSC2	**0.360**	**0.342**	0.195	**0.474**	**0.647**	-0.107
NCSC3	**0.285**	0.147	-0.040	**0.426**	**0.580**	-0.086
NCSC4	0.018	**0.356**	**0.311**	**0.271**	-0.007	0.060
NCSC5	0.029	**0.270**	0.187	**0.423**	**0.579**	-0.081

NCSC6	0.208	**0.396**			**0.578**	-0.008
NCSC7	**0.359**	-0.042			0.213	0.152
NCSC8					0.201	0.079
OPSC1	-0.176	**1.006**	-0.103	**0.431**	-0.264	**0.919**
OPSC2	**0.427**	0.168	0.077	**0.473**	0.302	**0.388**
OPSC3	0.097	**0.265**	-0.411	**0.719**	-0.190	**0.659**
OPSC4	**0.500**	-0.012	**0.310**	-0.041	0.210	**0.372**
OPSC5	-0.195	**0.669**	0.124	0.150	-0.045	**0.700**
OPSC6	-0.047	**0.441**	0.062	**0.464**	0.174	**0.552**
OPSC7	0.057	0.182	0.137	**0.316**	0.035	**0.518**
OPSC8			0.071	0.224		
OPSC9			-0.036	**0.551**		

Table 5: Promax Rotated Factor Loadings Number Concepts/Operations – Three Factor Solution, Form C

	Form C		
Item	F1	F2	F3
NCKC1	**0.625**	-0.066	0.018
NCKC2	**0.355**	-0.062	**0.284**
NCKC3	**0.240**	0.041	-0.038
NCKC4	**0.242**	**0.316**	-0.097
NCKC5	**0.558**	-0.137	0.102
NCKC6	**0.614**	-0.096	-0.201
NCKC7	**0.783**	0.021	-0.250
OPKC1	**0.725**	0.046	-0.103
OPKC2	**0.355**	-0.010	0.112
OPKC3	**0.393**	0.041	0.044
OPKC4	-0.131	0.046	**0.894**
OPKC5	**0.344**	0.064	0.112
OPKC6	**0.702**	0.067	-0.124
OPKC7	**0.571**	0.009	-0.211
OPKC8	**0.299**	**0.352**	0.045
NCSC1	-0.074	-0.008	**0.543**
NCSC2	**0.624**	-0.098	0.016
NCSC3	**0.462**	-0.200	**0.304**
NCSC4	-0.276	-0.177	**0.661**
NCSC5	**0.526**	-0.097	0.091
NCSC6	**0.499**	-0.072	0.197

NCSC7	0.098	0.046	**0.302**
NCSC8	0.167	0.044	0.096
OPSC1	-0.095	**0.899**	-0.129
OPSC2	**0.271**	**0.320**	0.157
OPSC3	-0.092	**0.655**	-0.070
OPSC4	0.243	**0.363**	-0.004
OPSC5	-0.107	**0.564**	0.303
OPSC6	0.131	**0.462**	0.210
OPSC7	0.003	**0.439**	0.183

Table 6: Percent communality explained by general and specific factors

		Percent of Communality Explained					
Form A	Scale	NC-KC	OP-KC	PFA-KC	NC-SC	OP-SC	TOTAL
	General	94.06%	76.23%	54.27%	67.96%	53.17%	66.98%
	Specific	5.94%	23.77%	45.73%	32.04%	46.83%	33.02%
Form B							
	General	87.33%	75.62%	72.88%	90.06%	60.13%	77.53%
	Specific	12.67%	24.38%	27.12%	9.94%	39.87%	22.47%
Form C							
	General	71.00%	80.61%	74.99%	62.52%	60.53%	70.37%
	Specific	29.00%	19.39%	25.01%	37.48%	39.47%	29.63%

Table 7: Reliabilities and Points of Maximum Information

Form A - Scale	N Items	Alpha	IRT reliability	Max Info
Number Concepts – Knowledge of Content	13	0.536	0.654	-0.51
Operations – Knowledge of Content	13	0.617	0.709	-0.21
Patterns, Function, Algebra – Knowledge of Content	12	0.740	0.771	-0.79
Number Concepts – Students and Content	10	0.494	0.576	-0.67
Operations – Students and Content	10	0.450	0.534	-1.97
Combined Number and Operations Knowledge of Content	26	0.719	0.810	-0.36
Combined Number and Operations Knowledge of Students and Content	20	0.622	0.709	-1.90
Total	58	0.845	0.907	-0.76
Form B - Scale	**N Items**	**Alpha**	**IRT reliability.**	**Max Info**
Number Concepts – Knowledge of Content	13	0.670	0.741	-1.45
Operations – Knowledge of Content	11	0.568	0.655	-0.76
Patterns, Function, Algebra – Knowledge of Content	12	0.793	0.805	-1.21
Number Concepts – Students and Content	8	0.507	0.578	-0.50
Operations – Students and Content	11	0.544	0.610	-1.29

Combined Number and Operations Knowledge of content	24	0.766	0.831	-1.27
Combined Number and Operations Knowledge of Student Thinking	19	0.657	0.727	-1.16
Total	55	0.878	0.916	-1.33
Form C Scale	N Items	Alpha	IRT reliability	Max Info
Number Concepts – knowledge of content	11	0.653	0.742	-0.95
Operations – knowledge of content	12	0.675	0.758	0.21
Patterns, Function, Algebra – knowledge of content	10	0.824	0.801	-0.81
Number Concepts – students and content	11	0.552	0.655	-0.43
Operations – students and content	10	0.649	0.689	-1.51
Combined Number and Operations Knowledge of Content	23	0.784	0.839	-0.17
Combined Number and Operations Knowledge of Student and Content	21	0.698	0.781	-1.11
Total	54	0.888	0.931	-0.92

References

Agresti, A. (1990) Categorical Data Analysis. New York: Wiley.

Author (2002). Measuring pedagogical content knowledge: An integrated measurement methodology. Unpublished manuscript, [name of university]

Author. (2002). ORDFAC software.

Ball, D. L. (1988). Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education. Unpublished doctoral dissertation, Michigan State University.

Ball, D.L. (1990). The mathematical understandings that prospective teachers bring to teacher education. Elementary School Journal, 90(4), 449-466.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter part of the equation. In J. Brophy (Ed.), Advances in research on teaching (Vol. 2, pp. 1-48). Greenwich, CT: JAI Press.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), Multiple perspectives on the teaching and learning of mathematics (pp. 83-104). Westport, CT: Ablex.

Ball, D. L., & Bass, H. (2001). Making mathematics reasonable in school. In G. Martin (Ed.), Research compendium for the Principles and Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics.

Ball, D.L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), Teaching as the learning profession: Handbook of policy and practice (pp. 3-32). San Francisco: Jossey-Bass.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), Handbook of research on teaching (4th ed.) (pp. 433-456) New York: Macmillan.

Bock, R. D. & Aitkin, M. (1981). Marginal maximum likelihood estimation of item parameters: application of an EM algorithm. Psychometrika, 46, 443-459

Cohen, D. K., & Ball, D. L. (1999). Instruction, capacity, and improvement (CPRE Research Report RR-043). Philadelphia, PA: Consortium for Policy Research in Education.

Bruner, J. (1960). The process of education. Cambridge: Harvard University Press.

Grossman, P. L. (1990). The making of a teacher : Teacher knowledge and teacher education. New York: Teachers College Press.

Hambleton, R. K. Swaminathan, H. & Rogers, H.J. (1991). Fundamentals of item response theory. Newbury Park, CA: Sage Publications.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. Journal of Educational Psychology, 77(3), 247-271.

Ma, L. (1999). Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States. Mahwah, NJ: Lawrence Erlbaum Associates.

Madfes, T., Montell, F., & Rosen, R. (2002). Independent evaluator: California Professional Development Institutes, Mathematics. San Francisco, CA: WestEd

Mislevy, R. J., & Bock, R.D. (1997). Bilog: Item analysis and test scoring with binary models. Scientific Software International.

National Commission on Teaching and America's Future (1996). What matters most: Teaching and America's Future. New York: Author.

Schwab, J. J. (1978). Education and the structure of the disciplines. In I. Westbury & N. Wilkof (Eds.), Science, curriculum, and liberal education: Selected essays (pp. 229-272). Chicago: University of Chicago Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

Schulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. Harvard Educational Review, 57, 1-22.

U.S. Department of Education (2002). Meeting the highly qualified teachers challenge: The Secretary's annual report on teacher quality. Washington, DC: U.S. Department of Education, Office of Postsecondary Education, Office of Policy, Planning, and Innovation.

Wilson, S. M., Shulman, L. S., & Richert, A. (1987). 150 different ways of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), Exploring teacher thinking, pp. 104-124. Sussex: Holt, Rinehart & Winston.

Wilson, S. W. & Wineburg, S. (1988). Peering at history through different lenses: the role of disciplinary perspectives in teaching history. Teachers College Record, 89, 525-39.

Appendix A

ITEMS

1. Mr. Allen found himself a bit confused one morning as he prepared to teach. Realizing that ten to the second power equals one hundred ($10^2 = 100$), he puzzled about what power of 10 equals 1. He asked Ms. Berry, next door. What should she tell him? (Mark (X) ONE answer.)

a) 0

b) 1

c) Ten cannot be raised to any power such that ten to that power equals 1.

d) -1

e) I'm not sure.

2. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

3. Mr. Fitzgerald has been helping his students learn how to compare decimals. He is trying to devise an assignment that shows him whether his students know how to correctly put a list of decimals in order of size. Which of the following sets of numbers will best suit that purpose?

a) .5 7 .01 11.4

b) .60 2.53 3.14 .45

c) .6 4.25 .565 2.5

d) Any of these would work well for this purpose. They all require the students to read and interpret decimals.

4. Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students focused on particular difficulties that they are having with adding columns of numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

I)	II)	III)
1	1	1
38	45	32
49	37	14
+ 65	+ 29	+ 19
142	101	64

Which have the same kind of error? (Mark ONE answer.)

a) I and II

b) I and III

c) II and III

d) I, II, and III